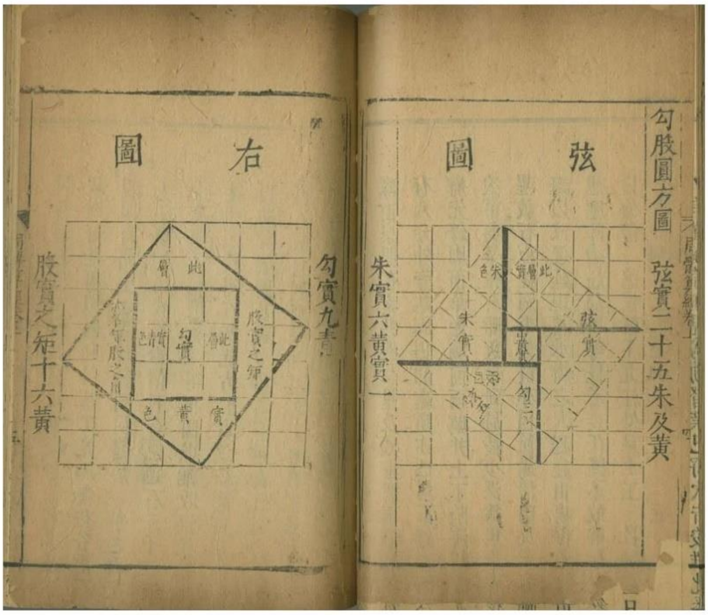
1. Suppose you have a little rubber square. If the square was simply moved from A to B, that change could most simply be expressed by an arrow connecting A and B, in other words, by a vector. But suppose the square is not moved, but instead the two opposite corners are pulled apart, resulting in a little parallelogram. Can the change from square to parallelogram represented in terms of:

a) a scalar b) a vector c) neither ?

Questions 2 to 7 inclusive are to be done as a group project

2. There are very many different proofs of Theorem 47 in Euclid's *Elements* (the Pythagorean Theorem) – they are of varying intricacy and some proofs are geometrical, while others algebraic. Shown here is a reproduction of a proof dating from 100 BC (Euclid's *Elements* itself dates from around 300 BC).



Two pages from the Zhou Bi Suan Jing *(Arithmetical Classic of the Gnomon and the Circular Paths of Heaven), an ancient Chinese book on astronomy and mathematics dating from approximately 100 BC - the page on the right demonstrates the Pythagorean Theorem. (The image is reproduced from a Ming dynasty copy printed in 1603.)*

Look up some of the proofs of the Pythagorean theorem in the relevant literature and reproduce at least one such proof that you find appealing.

3. In the context of Euclidean geometry,

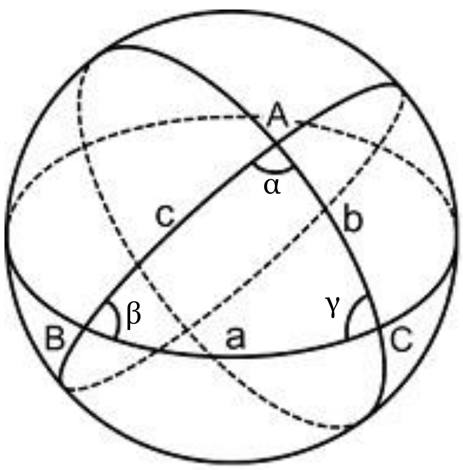
a) **Playfair's axiom is equivalent to Euclid's 5th or *parallel* postulate. Can you prove their equivalency?** [*Hint: You would need to show that Euclid's 5th postulate implies Playfair's axiom <u>and</u> that Playfair's axiom implies Euclid's 5th postulate.*]

b) **Assuming that the three angles of any triangle add up to 180°, show that Playfair's axiom (or equivalently, Euclid's 5th postulate) must be true.**

4a. Consider the surface of a sphere of radius R. On its surface, you can draw the *geodesic* triangle ABC (in other words, each side of the triangle is an arc of a great circle). Let the angles at A, B and C be α, β and γ (in radians) with opposite sides a, b and c respectively.

In spherical geometry, the usual law of cosines is instead:

$$\cos(c/R) = \cos(a/R)\cos(b/R) + \sin(a/R)\sin(b/R)\cos\gamma \text{ etc.}$$



Assume that just one of the angles (say, γ) of the *geodesic* triangle is a right-angle (in which case, note that the sum of the other two angles of the triangle will add to more than 90°).

Show that such a *geodesic* triangle drawn on the surface of a sphere does not satisfy the usual (Euclidean) Pythagorean theorem.
[*Hint: Use the Maclaurin series expansion for cosines:* $\cos x = 1 - x^2/2! + x^4/4! - \ldots$.]

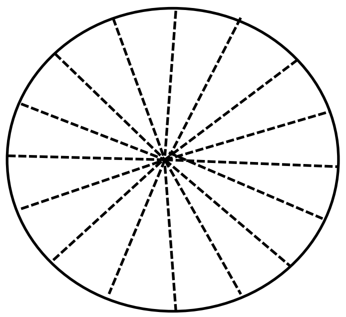
For a generalisation of the Pythagorean theorem to spherical geometry, see https://demonstrations.wolfram.com/SphericalPythagoreanTheorem/

4b. Say point A is the north pole, while B and C lie on the equator; take $\beta = \gamma = \pi/2$ and show that the area of the triangle ABC therefore equals αR^2. This is a special case of the formula, first obtained by the mathematician Thomas Harriot (in 1603!), for the area $A(\Delta)$ of any spherical triangle as a product of the angular excess $\mathcal{E}$ (the sum of its three angles less π) and the squared radius of the sphere:

$$A(\Delta) = (\alpha + \beta + \gamma - \pi)\, R^2 = \mathcal{E}(\Delta)\, R^2$$

Can you derive this expression?

5. You will need a spherical object (*eg*. a football or a basketball) and a large enough sheet of paper. First, measure the circumference (*ie.* the equator of the ball). Then, on the 'flat' sheet of paper, draw a circle with a radius that is one quarter of the ball's measured circumference. Cut along the circumference so that you now have a circular sheet of paper. Next, divide the circle you have drawn into sixteen equal wedges and cut along the dotted lines (being careful to keep the wedges joined at the centre of the circle) as indicated below.



a) Now try to *smoothly* (*i.e.* the pieces of paper should be made to lie flat on the surface of the ball) cover the northern hemisphere of the football with your circular sheet of paper, without leaving any gaps between the wedges – **are you able to do this? If not, explain why you think this is the case, and figure out a way that allows you to cover the northern hemisphere of the ball smoothly with the wedges of paper.**

In fact, the total arc length of the 'flat' circle's circumference which you will need to remove equals the amount of shortening required to obtain precisely the circumference of the equatorial circle on the ball's surface with a 'radius' given by the *geodesic* distance between the north pole and equator.

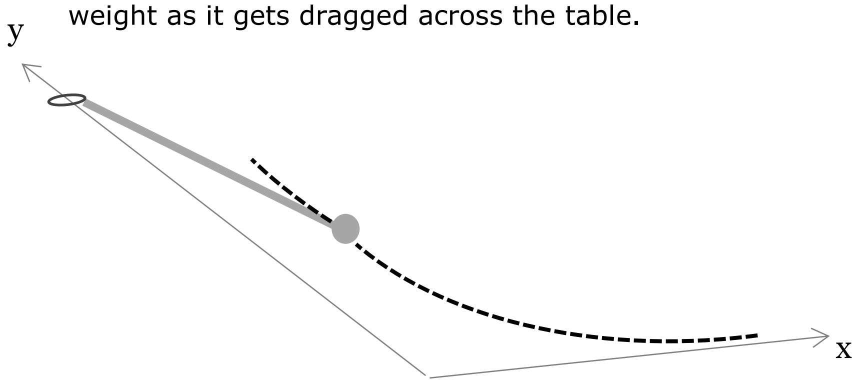
b) Geometrically speaking, your football is a surface of *constant positive curvature*, and one can get a good estimate for the *curvature* at any point on the ball's surface using the following expression:

$$\text{curvature} = \frac{\text{fractional circumference discrepancy} \times 6\pi}{\text{area responsible for the discrepancy}} \quad (\text{in radians/cm}^2)$$

What is your value for the curvature of the football?

Finally, assuming that the earth is a perfect sphere, estimate the *curvature* of its surface.

6. A curve known as the *tractrix* can be obtained by attaching a rod (or string) of fixed length R to a small weight and then laying the rod and weight flat on a table along the direction x (refer to the diagram below left) and then slowly pulling the free end of the rod (or string, keeping it taut) in a direction y that is perpendicular to the x-axis. A *tractrix* is the dotted curve traced out by the weight as it gets dragged across the table.



The surface obtained by rotating the tractrix around the y-axis is known as a *tractroid* (refer to the diagram on the right) and provides a model of a *pseudo-sphere* (a surface of constant negative curvature).

Note that, no matter the position of the dragged weight, the rotating rod always traces out a cone (tangent to the *pseudo-sphere*) of fixed slant length R. You can use this fact to **construct a physical model of a *pseudo-sphere* by the following procedure.**

i) Stack together as many sheets of paper as you can cut with a pair of scissors and staple them together along three edges (only a few staples per edge).

ii) Find the largest plate of bowl that fits within the size of your paper and trace out its circular edge.

iii) Next, cut along this circle to produce identical discs and repeat the procedure until you have at least 20 such discs – the more, the better!

iv) Take one of the discs and make a cut along its radius from a point on its circumference to the centre. Then slightly overlap the two sides of the cut and attach them (either glue, staple or use scotch-tape) together to create a very shallow cone.

v) Repeat the last step using another disc, but this time overlap the two sides a little more to create a slightly less shallow cone, but still with exactly the same slant length as your first cone. Place the new cone over the first one.

vi) Keep on repeating the last couple of steps, until you have used up all the discs which you had cut out.

You now have your own model *pseudo-sphere*!

7. For a general surface, an explicit formula that relates total curvature within a geodesic triangle to the excess/deficit from π of the sum of the three angles (θ_i with i=1,2,3) of the triangle is:

$$\int_{\Delta} K \, dA = \mathcal{E}(\Delta) \equiv \sum_i \theta_i - \pi$$

where K is the *Gaussian* or *intrinsic curvature* at any point within the triangle.

The special case of a *homogenous* space (a space which is the same at every point) is especially simple - such a space will have constant curvature. By considering an appropriate geodesic triangle on each of the following 2-dimensional homogenous spaces (in other words, surfaces of constant curvature) – the Euclidean plane, the surface of a sphere and *pseudosphere* (a surface with hyperbolic geometry), **use the formula above to determine the *Gaussian curvature* of each of these surfaces.**

What is the maximum area that a triangle satisfying hyperbolic geometry can have?

[*Hint: For this question you can use Lambert's formula for the area of a triangle in either spherical or hyperbolic geometry:*

$$\text{Area of triangle} = \pm\,(\theta_1 + \theta_2 + \theta_3 - \pi)\,R^2$$

(the + sign is for the sphere of radius R, while the – sign is for a pseudosphere with an equatorial or cusp radius of R).]

Optional Topic - a mathematically more sophisticated discussion followed by a question:

The Pythagorean theorem, Gauss' *theorema egregium* & the *metric tensor*

Consider a 3-dimensional cartesian co-ordinate system labelled by (x, y, z) and two points separated by a distance $\Delta\ell$. Then the Pythagorean theorem in 3-dimensions gives

$$\Delta\ell^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

Note that, if the two points are taken to be very close, then (in the usual notation of the infinitesimal calculus) we could set $\Delta\ell = d\ell$ *etc.* and instead write

$$d\ell^2 = dx^2 + dy^2 + dz^2$$

Now, let's write the Pythagorean theorem in another way. To do this we introduce g_{ab}, a so-called *second rank tensor*. (A *tensor* is a geometrical entity that is independent of a co-ordinate system. A scalar is a *zero rank tensor* and has a single value at every point in space. A vector is a *first rank tensor* and has three components at each point in 3-dimensional space.)

Using g_{ab}, the Pythagorean theorem in 3-dimensions becomes

$$d\ell^2 = \sum_{a,\,b\,=\,1}^{3} g_{ab}\, dx^a\, dx^b$$

With the notation that $dx^1 = dx$, $dx^2 = dy$ and $dx^3 = dz$. We may simplify this further by adopting the *summation convention* that repeated indices are to be summed over their range and write:

$$d\ell^2 = g_{ab}\, dx^a\, dx^b$$

Expanding the right-hand side of the last equation, we have:

$$d\ell^2 = g_{11}\, dx\, dx + g_{12}\, dx\, dy + g_{13}\, dx\, dz$$
$$+\ g_{22}\, dy\, dy + g_{21}\, dy\, dx + g_{23}\, dy\, dz$$
$$+\ g_{31}\, dz\, dx + g_{32}\, dz\, dy + g_{33}\, dz\, dz$$

Setting $g_{11} = g_{22} = g_{33} = 1$ and all the other g_{ab} values to zero, we recover the relation

$$d\ell^2 = dx^2 + dy^2 + dz^2$$

which is just the Pythagorean theorem in cartesian co-ordinates. In sum, we can calculate the distance between two nearby points in space as the *metric equation*

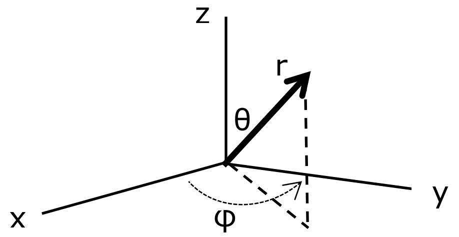
$$d\ell^2 = g_{ab}\, dx^a\, dx^b \qquad (1)$$

where the *metric tensor* g_{ab} (or *metric* for short) written out as a matrix is:

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2)$$

The metric tensor is extremely important in geometry because it enables us to calculate the distance between two points if we have the co-ordinate differences between the points. If we know the metric tensor of a space, we can construct most things that we might wish to know about that space. Note that, if a general space (called a *manifold* by mathematicians) does not possess a *metric*, then 'distance' has no meaning in that space. G*eo-metry,* originally derived from the words referring to "earth" and to "measure", requires the existence of a metric to make such distance measurements possible!

Equation (2) expresses the components of the metric tensor of 3-dimensional space within a cartesian co-ordinate system. However, the physically significant quantity we are measuring is the distance between two points and (as we shall see) in Newtonian physics, this distance is *invariant* and we must get the same answer no matter which co-ordinate grid we choose to work with. For example, we can re-write the Pythagorean theorem using instead spherical polar co-ordinates (r, θ, φ),



where (show this!) $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$ and $z = r \cos\theta$ (with $0 \le r \le \infty$, $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$). Then

$$d\ell^2 = dr^2 + r^2(d\theta^2 + \mathrm{Sin}^2\theta\, d\varphi^2) \qquad (3)$$

Equation (1) remains valid, but the components of the *metric tensor* now become

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\,\mathrm{Sin}^2\theta \end{pmatrix} \qquad (4)$$

We can if we wish use a completely arbitrary co-ordinate grid; while the components of the metric tensor will change from their cartesian values, the distance $d\ell^2$ however, will always be given by equation (1) above. This equation is more elegant and general than the usual form of the Pythagorean theorem, but this is at the expense of introducing a *metric tensor* with components that depend on the co-ordinate system being used. The approach might seem overly formal, with few advantages, so why go through all this trouble? It was in fact Gauss who first showed the importance of this approach to geometry.

Tasked with making a geodetic survey of the province of Hannover, Gauss was faced not with small, relatively flat areas to survey but with a large-scale region with hills and valleys.

He realised that the standard Pythagorean theorem with $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ was of little use to him. Instead, Gauss discovered a profound new result that he called the *theorema egregium* or 'outstanding theorem'. His theorem demonstrated that surveyors could lay down any arbitrary co-ordinate grid they wished to use on top of a surface – different surveyors were free to lay down different co-ordinate grids – and they could determine the shape of the surface from the manner in which the metric components of g_{ab} varied from place to place on the surface. In other words, what was important was not the co-ordinate grid itself (surveyors could use cartesian co-ordinates, polar co-ordinates, or any other system) nor the particular values of the metric components (since these would be different in different grids) but the variation with position of the metric components. If the 2-dimensional surface was covered with general co-ordinates (x^1, x^2) then the *theorema egregium* showed that all surveyors would agree on a quantity *K* known as the *(Gaussian) curvature.* The quantity *K* is thus an intrinsic property of the space The mathematical form of *K* (for a 2-dimensional space* and with an orthogonal metric**) is:

$$K = \frac{1}{2\, g_{11}\, g_{22}} \left\{ -\frac{\partial^2 g_{11}}{\partial (x^2)^2} - \frac{\partial^2 g_{22}}{\partial (x^1)^2} + \frac{1}{2g_{11}} \left(\frac{\partial g_{11}}{\partial x^1} \frac{\partial g_{22}}{\partial x^1} + \left(\frac{\partial g_{11}}{\partial x^2} \right)^2 \right) + \frac{1}{2g_{22}} \left(\frac{\partial g_{11}}{\partial x^2} \frac{\partial g_{22}}{\partial x^2} + \left(\frac{\partial g_{22}}{\partial x^1} \right)^2 \right) \right\} \tag{5}$$

* Surfaces are parametrised by two co-ordinates x^1 and x^2; consequently, when taking a derivative, we need to distinguish between it being with respect to x^1 or to x^2. This is done by taking a *partial derivative* (denoted $\partial/\partial x^1$ or $\partial/\partial x^1$) which simply means that when differentiating with respect to x^1, the other variable x^2 is to be kept fixed and vice versa.

** An orthogonal metric is one in which $g_{ab}=0$ for $a \neq b$ which implies that the co-ordinate axes for x^1 and x^2 cross at right-angles, as is the case for the metric tensors we are considering. The general expression for *K,* involving non-diagonal components of g_{ab}, is a little more complicated.

Finally, the question:

<u>Question</u>: Using Gauss' *theorema egregium, i.e.* equation (5) above, show that $K = 1/R^2$ for the surface of a sphere of radius R.

[*Hint: Use the metric equation in spherical polar co-ordinates, (3) & (4) above, taking the r co-ordinate to have the constant value R as the (fixed) radius of the sphere. Then identify the variables (x^1, x^2) in equation (5) with the two remaining co-ordinates (θ, φ).*]